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ON SHOCKWAVE PROPAGATION IN AN ELASTIC SPACE WITH FINITE DEFORMATIONS

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A. D. CHERNYSHOV
(Voronezh)
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The influence of finiteness of the deformations and of the convective terms, in determining the medium velocity in terms of the displacements, on shockwave propagation in a three-dimensional elastic medium is investigated. The Almansi tensor [1] is utilized as the finite strain tensor. It is found that the quantity of shocks and their properties depend strongly on the deformations of the medium ahead of the surface of strong discontinuity, and on whether or not nonlinear terms in the rheological equations are taken into account. Thus, propagation of three different shocks is possible in the case of small deformation when these equations are written exactly. The particular case when the medium is in the undeformed state ahead of the shock is singular: all the qualitative results agree with the results of the analogous linear problem. Expressions for the shock velocities are obtained explicitly in particular cases.

1. Let us write the connection between the stress tensor σ_{ij} and the Almansi finite strain tensor e_{ij} as

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \quad e_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i} - \alpha u_{k,i}u_{k,j}) \tag{1.1}$$

where λ and μ are the Lamé coefficients, u_i the displacements of the medium particles.

The coefficient α equals unity if finiteness of the strains is taken into account, and $\alpha = 0$ if the small strain tensor is utilized. Introduction of the coefficient α permits clarification of the influence of finiteness of the deformation of the medium on shockwave propagation.

The governing equation (1.1) is valid in a first approximation for small deformations of the medium when the elastic potential W has the form

$$W = {}^1/_2\lambda\,(e_{kk})^2 + \mu e_{ij}e_{ij}$$

Let us consider propagation of a surface of strong discontinuity Σ in an elastic medium. To simplify the calculations, let us introduce a moving rectangular coordinate system such that its origin would move together with the surface of discontinuity at a velocity G. Let us direct the x_3-axis at an arbitrary material point on Σ under consideration along the normal to this surface, then the x_1- and x_2-axes will be in the tangent plane to the surface of discontinuity. Let the Greek subscripts α, β, ... take the values 1 or 2, and the Latin subscripts i, j, k, ... the values 1, 2 or 3. We shall calculate all the quantities in a fixed coordinate system and project them on the axes of the moving system. To pass from the fixed to the moving coordinate system we write the relationships

$$\frac{\partial}{\partial x_i} = \delta_{i3}\frac{\partial}{\partial x_3} + \delta_{i\alpha}\frac{\partial}{\partial x_\alpha}, \qquad \frac{\partial}{\partial t} = \frac{\delta}{\delta t} - G\frac{\partial}{\partial x_3} \tag{1.2}$$

Here $\delta/\delta t$ is the delta derivative with respect to time [2]. Since the displacements are continuous on Σ, then only the components $u_{i,3}$ of the tensor components $u_{i,j}$ will be discontinuous, i. e.

$$[u_{i,j}] = [u_{i,3}]\,\delta_{j3} \tag{1.3}$$

The square brackets in (1.3) denote jumps in the discontinuous quantities on the shock. The jumps $[u_{i,3}]$ are connected with the jumps in the particle velocity v_i of the medium. We find this connection from the dependence

$$v_i = \frac{\delta u_i}{\delta t} + (\beta v_3 - G)\,u_{i,3} + \beta v_\alpha u_{i,\alpha} \tag{1.4}$$

which is the definition of the velocity in terms of the displacements in the moving coordinate system. The coefficient β in (1.4) is introduced for the same purpose as the coefficient α in (1.1). When $\beta = 1$, then convective terms are kept in (1.4); if these terms are neglected, it is necessary to put $\beta = 0$. Evaluating the jump in (1.4) we obtain

$$[v_i] = \beta u_{i,\alpha}\,[v_\alpha] + [(\beta v_3 - G)\,u_{i,3}] \tag{1.5}$$

Let us examine the special case of deformability of a medium ahead of a shock when $u_{k,\alpha}^+ = 0$ and $u_{k,3}^+ \neq 0$.

Only $[e_{kk}]$ and $[e_{i3}]$ of the jumps in the tensor e_{ij} are needed, and we find expressions for these from (1.1) and (1.3)

$$[e_{kk}] = [u_{3,3}] - \alpha u_{k,3}^+\,[u_{k,3}] + {}^1/_2\alpha\,[u_{k,3}]\,[u_{k,3}] \tag{1.6}$$

$$[e_{i3}] = {}^1/_2\,[u_{i,3}] + {}^1/_2\,[u_{3,3}]\,\delta_{i3} - \alpha u_{k,3}^+\,[u_{k,3}]\,\delta_{i3} + {}^1/_2\alpha\,[u_{k,3}]\,[u_{k,3}]\,\delta_{i3}$$

Setting $j = 3$ in the governing equation (1.1), we obtain in the discontinuities

$$[\sigma_{i3}] = \lambda\,[e_{kk}]\,\delta_{i3} + 2\mu\,[e_{i3}] \tag{1.7}$$

To close the system of equations (1.5)–(1.7) relative to the jumps in the discontinuous quantities, let us write the dynamical compatibility conditions of the discontinuities on the shock [2]

$$[\sigma_{i3}] = \rho^- (\beta v_3^- - G) [v_i], \quad [\rho (\beta v_3 - G)] = 0 \tag{1.8}$$

Let us project (1.7) and (1.8) onto the tangent plane to Σ by setting $i = \alpha$; we obtain

$$[\sigma_{\alpha 3}] = \mu [u_{\alpha,3}] = \rho^- (\beta v_3^- - G) [v_\alpha] \tag{1.9}$$

The equalities (1.9) say that the vectors $[v_i]$ and $[u_{i,3}]$ lie in one plane with the normal to the surface Σ. Let us call this the characteristic plane. Let us rotate the x_1 coordinate system around the x_3-axis so that the x_i-axis would lie in the characteristic plane. Let y_i denote the new coordinate system. The angle of rotation φ equals the angle between the x_1- and y_1-axes, and the transformation matrix $L = \| l_{ij} \|$ has the components [1]

$$l_{11} = l_{22} = \cos\varphi, \quad l_{21} = - l_{12} = \sin\varphi, \qquad l_{i3} = l_{3i} = \delta_{i3} \tag{1.10}$$

All the considerations are presented below in the new coordinate system. We denote the quantities v_i, $u_{i,3}$ and the others in the y_i system without primes, while they will be primed in the x_i system.

The position of the y_i coordinate system is unknown, and is determined by the angle φ, hence the components of the tensor $u_{i,j}$ are also unknown. If the components $u'_{i,j}$ of this tensor are known in the x_i coordinate system, then the tensor $u_{i,j}$ is expressed by using the matrix L and applying the formula

$$u_{i,j} = l_{ki} l_{nj} u_{k,n} \tag{1.11}$$

In order to find φ, we find from (1.9)

$$[v_2] = [u_{2,3}] = 0 \tag{1.12}$$

Taking account of (1.12), the velocity jumps $[v_i]$ from (1.5) are expressed in terms of $[u_{i,3}]$ as follows:

$$[v_1] \Delta = (\beta v_3^- - G) \{\beta u_{1,3}^+ [u_{3,3}] + (1 - \beta u_{3,3}^+) [u_{1,3}]\} \tag{1.13}$$

$$[v_3] \Delta = (\beta v_3^- - G) [u_{3,3}], \qquad \Delta = 1 - \beta u_{3,3}^+$$

Besides (1.13), still another equation can be obtained from the three relationships (1.5) by setting $i = 2$

$$\beta u_{2,3}^+ [v_3] = 0 \tag{1.14}$$

For $\alpha = 1$ we obtain from (1.9) and (1.13)

$$\mu \Delta [u_{1,3}] = V \{\beta u_{1,3}^+ [u_{3,3}] + (1 - \beta u_{3,3}^+) [u_{1,3}]\} \tag{1.15}$$

$$V = \rho^- (G - \beta v_3^-)^2$$

for $i = 3$ we find from (1.7) and (1.8)

$$(\lambda + 2\mu) \Delta \{(1 - \alpha u_{3,3}^+ + {}^1/_2 \alpha [u_{3,3}]) [u_{3,3}] - \alpha (u_{1,3}^+ - {}^1/_2 [u_{1,3}]) [u_{1,3}]\} = V [u_{3,3}] \tag{1.16}$$

Substituting (1.13) into (1.14) results in still another equation

$$\beta u_{2,3}^+ [u_{3,3}] = 0 \tag{1.17}$$

The identity $\beta \equiv \beta^2$ was taken into account in obtaining (1.17). The equation

$$[u'_{2,3}] \cos\varphi - [u'_{1,3}] \sin\varphi = 0 \tag{1.18}$$

can be obtained from (1.11) and (1.12).

Substituting (1.11) into (1.15)–(1.17), we obtain three equations which, together with (1.18), are equivalent to the system (1.15)–(1.18), and from which the unknown parameters φ, V and the jumps in two projections of the vector $[u_{i,3}]$ are found, while the remaining jump is considered given.

Let us note that (1.18) is a corollary of the second equality in (1.12) while (1.17) is a corollary of both equalities in (1.12). If we set $\beta = 0$ everywhere, then (1.17) will be satisfied identically. This is explained by the fact that both equalities in (1.12) result in the same equation for $\beta = 0$, which is (1.18), hence we obtain an undetermined system to find the unknown parameters.

The propagation velocity of a longitudinal shock is found from (1.16) only in the particular case with $[u_{\alpha,3}] = 0$ for a given jump $[u_{3,3}]$. In the remaining cases, the unknown parameters on the shock are determined under the condition of specifying either the jumps of two projections of the vector $[u_{i,3}]$ or the jump in one projection of this vector and specifying the angle φ.

Therefore, the problem of shockwave propagation in an elastic medium taking account of nonlinear convective terms in the determination of the velocity in terms of the displacements differs qualitatively from the same problem when these terms are discarded, independently of whether finiteness of the deformations is taken into account or not. Let us still note that (1.17) is satisfied identically also if the medium is in the undeformed state ahead of the front of the surface Σ. In both problems, when $\beta = 0$ or when $u_{i,j} = 0$ taking account of finiteness of the deformation is quantitative in nature.

2. For $\beta = 0$ or when the medium ahead of the shock front is in the undeformed state, we find from (1.15) and (1.16)

$$V_1 = (\lambda + 2\mu)(1 - \alpha u_{3,3}^+ + {}^1/_2\alpha [u_{3,3}]) \quad \text{for } [u_{1,3}] = 0 \tag{2.1}$$

$$V_2 = \mu \quad \text{for } [u_{1,3}] \neq 0 \tag{2.2}$$

For $[u_{3,3}]$ we have from (2.2) and (1.16):

for $\alpha = 0$

$$[u_{3,3}] = 0$$

for $\alpha = 1$

$$[u_{3,3}] = u_{3,3}^+ - \frac{\lambda+\mu}{\lambda+2\mu} - \left\{\left(u_{3,3}^+ - \frac{\lambda+\mu}{\lambda+2\mu}\right)^2 + (2u_{1,3}^+ - [u_{1,3}])[u_{1,3}]\right\}^{1/2} \tag{2.3}$$

The square root in (2.3) is taken with a minus sign since otherwise the jump $[u_{3,3}]$ would not tend to zero for $[u_{1,3}] \to 0$, and the shock would not pass into the appropriate sonic wave.

Let us examine the case when $\beta = 1$, $\alpha = 0$, and the elastic medium ahead of the surface Σ is in the deformed state. Let us consider $[u_{3,3}]$ given, and $[u_{\alpha,3}]$, V and φ to be found. After eliminating $[u_{1,3}]$, equations (1.15) and (1.16) simplify and become

$$\Delta^+(\mu - V)[u_{1,3}] = V u_{1,3}^+ [u_{3,3}], \qquad \{\Delta^+(\lambda + 2\mu) = V\}[u_{3,3}] = 0 \tag{2.4}$$

We hence obtain for the roots

$$\begin{aligned} V_1 = (\lambda + 2\mu)\Delta^+, \qquad & [u_{1,3}] = u_{1,3}^+[u_{3,3}], \qquad (\lambda + 2\mu)/\mu - V_1 \\ V_2 = \mu, \qquad & [u_{1,3}] \neq 0, \qquad [u_{3,3}] = 0 \end{aligned} \tag{2.5}$$

The angle φ defining the position of the characteristic plane for the first wave is found from (1.17). Substituting (1.11) into (1.17) and taking into account that $[u_{3,3}] \neq 0$ on the wave under consideration, we obtain the required equation

$$\operatorname{tg} \varphi = u_{2,3}^{+'}/u_{1,3}^{+'} \tag{2.6}$$

From (2.5) the quantity V_2 corresponds to a transverse shock. Taking into account that now $[u_{3,3}] = 0$ and (1.17) is satisfied identically, we find the angle φ from (1.18)

$$\operatorname{tg}\varphi = [u'_{2,3}]/[u'_{1,3}] \tag{2.7}$$

Therefore, propagation of two different shockwaves is possible in the approximation under consideration.

For an exact formulation of the problem when $\alpha = \beta = 1$, we obtain a cubic equation

$$(1 - u_{3,3}^{+}) V^3 - \{2\mu(1 - u_{3,3}^{+}) + (\lambda + 2\mu)(1 - u_{3,3}^{*})(1 - u_{3,3}^{+})^2 +$$
$$+ (\lambda + 2\mu)(1 - u_{3,3}^{*}) u_{1,3}^{+2} - \mu u_{1,3}^{+}\} V^2 + \Delta\{\mu^2 + \mu(\lambda + 2\mu)[u_{1,3}^{+2} + 2(1 -$$
$$- u_{3,3}^{*})(1 - u_{3,3}^{+})]\} V - \Delta^2\mu^2(\lambda + 2\mu)(1 - u_{3,3}^{*}) = 0 \tag{2.8}$$
$$u_{3,3}^{*} = {}^1/_2 (u_{3,3}^{+} + u_{3,3}^{-})$$

from (1.15) and (1.16) to find V.

If the term $u_{1,3}^{+2}$ is neglected in this equation, as can be done in the case of small deformations of the medium ahead of the shock, or when $u_{1,3}^{+} = 0$, then we obtain for the roots

$$V_1^{\circ} = (\lambda + 2\mu)(1 - u_{3,3}^{+})(1 - u_{3,3}^{*}), \qquad u_{1,3}] = 0, \qquad V_3^{\circ} = V_2^{\circ} = \mu, \ [u_{3,3}] = 0 \tag{2.9}$$

The first root corresponds to a longitudinal shockwave for which the angle φ is determined from (2.6). The coincident roots V_2° and V_3° correspond to a transverse shock for which the angle φ is found from (2.7). In an exact formulation of the problem when the term $u_{1,3}^{+2}$ is not discarded in (2.8), all three roots V_1, V_2 and V_3 are different, and the jumps $[u_{1,3}]$ and $[u_{3,3}]$ differ simultaneously from zero. Hence, after dividing by $[u_{3,3}]$ in (1.17), we arrive at (2.6), which will hold simultaneously for all three roots.

If $u_{1,3}^{+}$ is considered a small quantity in (2.8), the roots of this equation are written to higher order accuracy as

$$V_1 = V_1^{\circ} + V_1^{\circ}(\lambda + 2\mu)\,\frac{(\lambda + 2\mu)(1 - u_{3,3}^{*})^2 - \mu}{(V_1^{\circ} - \mu)^2}\, u_{1,3}^{+2} \tag{2.10}$$

$$V_2 = \mu \pm \mu u_{1,3}^{+}\left(\frac{(\lambda + 2\mu)[u_{3,3}]}{(1 - u_{3,3}^{+})(\mu - V_1^{\circ})}\right)^{1/2} \tag{2.11}$$

Terms containing $u_{1,3}^{+}$ to powers not higher than the second are kept in obtaining (2.10), and not higher than the first for (2.11). The inequality

$$|u_{1,3}^{+}| \ll |[u_{3,3}]|$$

was hence assumed.

For small deformations of the medium, the inequality $(1 - u_{3,3}^{+})(\mu - V_1^{\circ}) < 0$ is satisfied, hence there results from (2.11) that both shockwaves from (2.11) are possible under the condition

$$[u_{3,3}] \leqslant 0 \tag{2.12}$$

In general, the cubic equation (2.8) always has one real root, but the other two depend on the sign of the discriminant of this equation. The inequality (2.12) should be understood as the condition for positivity of the discriminant, which assures all three roots are real. This inequality is one of the consequences of the qualitative influence of the nonlinear terms. In the linear approximation, or when not all the nonlinear terms are taken into account, this condition is missing.

The propagation velocity of sound waves can be obtained from (2.8). To do this it is sufficient to set $u_{3,3}^{+} = u_{3,3}^{-} = u^{*}_{,3}$ and to omit the plus and minus signs everywhere

$$V_1 = V_2 = {}^1/_2\mu + {}^1/_2(\lambda + 2\mu)\{(1 - u_{3,3})^2 + u_{1,3}^2\} \pm {}^1/_2 D^{1/2}$$
$$D = \{\mu - (\lambda + 2\mu)(1 - u_{3,3})^2\}^2 + (\lambda + 2\mu)\{2(\lambda + 2\mu)(1 - u_{3,3})^2 + $$
$$+ 2\mu + (\lambda + 2\mu) u_{1,3}^2\} u_{1,3}^2, \quad V_3 = \mu \tag{2.13}$$

The propagation velocities of the sound waves are similar to the velocities of weak shockwaves. Therefore, weak shockwaves, as well as sound waves, in a three-dimensional elastic medium can propagate at three different velocities, one of which equals the transverse wave velocity in the linear approximation.

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ASYMPTOTIC ANALYSIS OF WAVE MOTIONS OF A VISCOUS FLUID WITH A FREE BOUNDARY

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E. N. POTETIUNKO and L. S. SRUBSHCHIK
(Rostov-on-Don)
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Asymptotic expansions for the solution of the Cauchy-Poisson problem of wave motion of a viscous incompressible fluid with infinite depth are constructed at large Reynolds numbers. A proof of the asymptotics is given. Examples of plane and spatial motions are presented in which the asymptotic expansion is determined in the form of a free surface.

In the case of plane motion a solution of this problem was obtained in closed form and was analyzed in some particular cases in [1] by the integral transformation method. The problem was solved by the same method in other papers also. A discussion of these papers is presented in [2].

Moiseev proposed the asymptotic method [3–7] for the solution of this and a number of other problems.

Theorems of existence and uniqueness for solutions of unsteady linearized Navier-Stokes equations for the motion of a viscous fluid with a free surface in an open vessel were obtained in papers [8–10] in the absence and presence of surface tension.

In this paper an asymptotic method is also proposed. However, the method used for finding the asymptotics leads to simpler and more convenient expressions for numerical analysis than in [1, 3].

In Sect. 2 asymptotic expansions of the solution at large Reynolds numbers are constructed with any arbitrary preassigned degree of accuracy. The construction of the asymptotics is carried out by the method presented in paper [11]. In this connection the first and second iteration processes are applied simultaneously to the equations and boundary conditions. As a result of this, the initial system at each stage decomposes into two independent problems for the potential and vortical parts of the motion.